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"The Quantization and Measurement of Momentum Observables II"

by

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Abstract

Concepts of classical and quantum global measurability introduced in an earlier paper (Wan and McFarlane 1980) are discussed in greater detail and in the context of an n -dimensional Riemannian manifold. The ideas are illustrated with diagrams and examples. A notion of exact measurability is introduced and is shown to imply quantum global measurability. The physically important Killing momenta are shown to be exactly and hence quantum globally measurable.

1: Introduction

This paper is a sequel to and a development of a recent paper (Wan and McFarlane 1980; hereinafter referred to as paper 1) in which we introduced the concepts of classical and quantum global measurability and motivated their study by showing that measurability, in either of the above senses, requires of a momentum that it be complete, and therefore that it be quantizable by means of geometric procedures (Mackey 1963, Wan and Viazminsky 1977, 1979). The present paper concerns itself with the concepts of classical and quantum global measurability in the context of an n -dimensional Riemannian manifold, their development, definition, and illustration by means of examples. A refinement of classical global measurability, termed exact measurability, is then introduced, shown to imply quantum global measurability, and moreover to be a property of the momenta associated with the physically important Killing vector fields.

2: Classical Momentum Measurement

The goal of this section is to construct the concept of classical global measurability in the context of a Riemannian manifold, and to relate it to the completeness of a momentum.

2.1: The measurement Process

The problem herein lies in the measurement of a momentum P corresponding to a vector field X on a complete Riemannian manifold M of metric G . Around every non-critical point of X there exists a local chart x^i in terms of which X assumes the form $\partial/\partial x^1$. In this chart P becomes the momentum p_1 conjugate to x^1 and the metric assumes the usual tensor form $g^{ij}(x^k)$.

In paper 1 we used the model of Aharanov and Safko (1975) to measure momenta and here we shall also use this model. The model involves a collision between two particles, one a test particle described by undashed quantities, and another a reference (measuring) particle described by dashed quantities. The Hamiltonian describing the collision is given by

$$H = \frac{1}{2m} g^{ij}(x) p_i p_j + \frac{1}{2m} g^{ij}(x') p'_i p'_j + \omega(t) g^{ij}(x, x') p_i p'_j, \quad (1)$$

where $g^{ij}(x, x')$ is the parallel propagator, and $\omega(t)$ is a function of t satisfying

$$\omega(t) = \begin{cases} \omega_0, & t \in (0, T), \\ 0, & t \notin (0, T). \end{cases} \quad (2)$$

The non-uniqueness of the parallel propagator introduced by Synge (1971) causes no difficulty as we shall consider only local motion. Local motion is most simply described in terms of a local chart. In particular we can, in the neighbourhood of the collision, elect a local Cartesian chart y^i such that (see paper 1)

$$y^1 = b(m_0)(x^1 - a^1), \quad (3)$$

where m_0 is the point given by $x^i = a^i$, that is m_0 is the origin of the chart y^i . In terms of y^i the Hamiltonian becomes

$$H = \frac{1}{2m} \eta^{ij} p_i^0 p_j^0 + \frac{1}{2m} \eta^{ij} p'_i p'_j + \omega(t) \eta^{ij} p_i^0 p_j^0, \quad (4)$$

where $\eta^{ij} = \delta^{ij}$ is the Cartesian metric, and p_i^0 is the momentum conjugate to y^i . Hamilton's equations of motion yield, upon taking the impulsive limit $\omega_0 T \rightarrow \gamma$, that $\gamma p_i^0 = \Delta y^i$. Whence we may deduce the equation connecting P , the test particle's momentum, and Δy^1 the measured displacement along y^1 as

$$P = p_1 = b(m_0) p_1^0 = \gamma^{-1} b(m_0) \Delta y'^1. \quad (5)$$

In the process the reference particle recoils by an amount and in a direction determined by the test particle's momentum. We can arrange the orientation of the reference particle prior to the interaction such that $p_i^0 = 0, i \neq 1$, resulting in a recoil $\Delta y'^1$ of the test particle along the y'^1 -axis, as is portrayed in figure 1. We shall adopt this option when employing the measuring procedure in the sequel. The parameter γ is assumed known.

2.2: On Local Measurement

Let A be an open neighbourhood of a point m_0 in the configuration manifold M satisfying the following two conditions:

(i) A is coverable by a single local chart x^i in terms of which the momentum P becomes p_1 conjugate to x^1 , and (ii) A is sufficiently small that the departure within A of M from a Euclidean space is negligible. Then expressions (3) and (4) are well defined in A . We now introduce the notion of local measurement. A momentum measurement is said to be executed locally in A if (i) no account is taken of position or momentum values outwith A , so that if a momentum value is to be determined within A by means of the impulsive collision described above then neither the test particle incident, nor the reference particle employed, can so recoil as to leave A ; and (ii) A must possess a certain physical size, that is it must contain an open sphere of some fixed radius d_0 , so as to be able to measure momenta corresponding to a given range of particle energies $E \leq E_A$. If the second condition is not satisfied then, for certain angles of incidence of the test particle, the reference particle will after collision so recoil as to leave A .

In executing a local measurement in A we encounter two sources of possible error, that in the measurement of the reference particle recoil $\Delta y'^1$ and therefore in the evaluation of p_1^0 , and that in the identification of the point m_0 in A from which P may be calculated using (5). We shall assume that the former source of error may be made vanishingly small, or to be more definite we shall assume an exact measurement of $\Delta y'^1$, or as is equivalent an exact measurement of p_1^0 . The latter source of error, however, has an intrinsic physical significance arising from the impulsive nature of the collision between test and reference particles which cannot be eliminated by perfection of the apparatus.

As depicted in figure 1 the test particle of momentum P recoils within the one-dimensional set B whose extent is determined solely by the parameter γ and the reference particle momentum p_1^0 . It is this finite (non-zero) extent of B which gives rise to an irreducible error; for it is impossible to say at which point m of B the momentum of the test particle was determined. We are faced, therefore, with the problem of defining the value attributed to P by a locally executed measurement in A , and of quantifying the error arising from the variability of $b(m)$ within B . We shall take the value of P as

$$P = b(B) p_1^0, \quad (6)$$

where

$$b(B) = \int_B b(y') dy' / \int_B dy',$$

which may be regarded as the mean value of $b(m)$ in B . For the error we shall take the standard deviation

$$\Delta_B P = \left(\int_B (p_1^0 b(y') - p_1^0 b(B))^2 dy' / \int_B dy' \right)^{1/2} = |p_1^0| \Delta_B b. \quad (7)$$

2.3: Global Measurability and Completeness

In paper 1 we introduced the concept of global measurability of P . We shall now explicitly generalize this notion to a Riemannian manifold. The aim is to establish a basically position-independent measuring process. The fundamental reason for this originates in quantum mechanical considerations, that is the measurement of a quantised momentum should be basically position-independent so as not to conflict with the uncertainty principle. One then examines whether quantisable classical momenta possess this property, and whether it is related to the quantisability condition for momenta. Now in our model of measurement of classical momenta the measuring device is local, consisting of the reference particle (characterised by its momentum p_1^0 conjugate to the local Cartesian coordinate y^1) which interacts with the measured particle with a certain fixed value of γ . We shall call two measuring devices sited in the neighbourhoods of two distinct points m_1, m_2 in M identical if their respective local properties at m_1 and m_2 are the same, that is if they possess the same values of p_1^0 and γ .

Returning to the momentum P of the test particle,

let $A_\Omega = \{A\}$ be an open covering of the maximal integral curve Ω of the vector field X associated with P . Here every A in A_Ω is assumed to satisfy the conditions described in section 2.2 so that a local measurement of P may be performed in A . Local measurements of P in A using identical measuring devices produce a common value γp_1^0 for the recoil Δy^1 whatever the position of the test particle in A_Ω i.e. values independent of the choice of A . In sharp contrast the error $\Delta_B P$ does depend on A . As we execute local measurements of P in A , moving along Ω and using identical measuring devices, $\Delta_B P$ may vary unboundedly, depending upon the nature of the momentum involved. This would imply that we cannot meaningfully measure P along Ω using identical measuring devices. In other words we cannot in general obtain a position independent measuring process.

We are led to the notion of global measurability: a momentum P is said to be classically globally measurable iff it is possible to elect, for each integral curve Ω of the vector field X associated with P , a finite upper bound $\Delta_\Omega P$ to the set of errors arising from locally executed measurements in $A \in A_\Omega$ along Ω using identical measuring devices. This concept, illustrated in figure 2, is a generalization of the idea of global measurability

†: Here we assume that the Riemannian manifold is well behaved (has no singular point) so that we could have a minimal size for A , i.e. every A contains an open sphere of some fixed common radius. This then allows the use of identical measuring devices along A_Ω .

given in paper 1[‡]. Mathematically all this, while clearly expressing physical intuition, is somewhat vague and ambiguous. It is necessary to give a precise mathematical definition. To do so we introduce families of local recoil sets. For each maximal integral curve Ω of X let $\{B\}_{\ell_\Omega}$ be the family of all connected arcs B of Ω of fixed metrical length ℓ_Ω . We call $\{B\}_{\ell_\Omega}$ a family of local recoil sets on Ω . Note that ℓ_Ω , while having a fixed value for a given Ω can vary between integral curves. We can now give the desired definition: a momentum P is classically globally measurable iff, for every integral curve Ω of X , it is possible to elect a finite upper bound Δ_Ω^P to the set of uncertainties

$\Delta_B^P, B \in \{B\}_{\ell_\Omega}$ (calculated with a pre-assigned value p_1^1 for all B) generated by the family of local recoil sets $\{B\}_{\ell_\Omega}$. This definition reflects our physical concept since for sufficiently small regions we can identify Δ_y^1 with the arc B (hence the name local recoil set for B).[†] The relevance of global measurability is seen in the following theorem whose proof is a simple extension of the proof for theorem 5 in paper 1.

‡: Different measuring devices are allowed for different integral curves.

†: The question arises as to whether such identification with a pre-assigned value of Δ_y^1 can be made everywhere along Ω . We may indeed make such an identification along Ω provided that the curvature K (Stoker 1969) of Ω is everywhere bounded. If there are regions in which K is unbounded then the simple identification of Δ_y^1 and B fails in these regions.

Theorem 1: On Completeness and Classical Global Measurability.

A momentum P is classically globally measurable only if it is complete.

3: Quantum Momentum Measurement

We address in this section the problem of the global measurability of a quantum momentum $Q(P)$, and shall begin by considering local measurement.

3.1: On Local Measurement

Let $C_0^\infty(A)$ and $C_0^\infty(M)$ be the infinitely differentiable functions of compact support in A and M respectively. Let $Q_0(P)$ be the symmetric operator in $L^2(M)$ given by

$$Q_0(P) = -i\hbar (X + \frac{1}{2} \text{div} X), \quad (8)$$

with domain

$$DQ_0(P) = \{\psi \in L^2(M) \mid \psi \in C_0^\infty(M), Q_0(P)\psi \in L^2(M)\}.$$

Then the quantum momentum $Q(P)$ whenever it exists, is simply (see paper 1) $Q_0^+(P)$. Now let x^i be a chart covering A in terms of which $P = p_1$, the momentum conjugate to x^1 . Naturally one tries to construct the quantum analogues, $Q(x^1)$ and $Q(p_1)$, of the classical canonical pair, (x^1, p_1) . However, neither $Q(x^1)$ nor $Q(p_1)$ can be simply defined as a self-adjoint operator in $L^2(M)$. The cause of the trouble is the fact that the chart x^i is non-global in general, so that x^1 is not well-defined outwith A . We should therefore seek operators in

$$L^2(M) \text{ with the property that their restrictions on } L^2(A)$$

correspond to x^1 and p_1 . Such operators may not be unique. This does not matter when one is dealing with problems purely in A and $L^2(M)$ as we shall be in this section. A natural choice of these operators is as follows: For $Q(x^1)$ we define

$$Q(x^1)\psi = x^1 \chi_A \psi, \quad \psi \in L^2(M) \quad (9)$$

where χ_A is the characteristic function of A ($\chi_A(m)=1, m \in A$, and $\chi_A(m)=0, m \notin A$). For $Q(p_1)$ we take

$$Q(p_1) = Q(P) = Q_0^\dagger(P), \quad (10)$$

whenever $Q(P)$ exists. Confining ourselves to $C_0^\infty(A)$ we have the following well-defined expressions:

$$Q(p_1)\phi = Q_0(P)\phi = -i\hbar \left(\partial/\partial x^1 + \frac{1}{2} \partial(\ln \sqrt{g})/\partial x^1 \right) \phi, \quad (11)$$

$$[Q(x^1), Q(p_1)]\phi = [Q(x^1), Q_0(P)]\phi = -i\hbar \phi, \quad (12)$$

$$\Delta Q(x^1) \Delta Q(p_1) \geq \hbar/2, \quad \Delta Q(x^1) \Delta Q_0(P) \geq \hbar/2, \quad (13)$$

where $\phi \in C_0^\infty(A)$ and $\Delta Q = [\langle \phi | Q^2 | \phi \rangle - \langle \phi | Q | \phi \rangle^2]^{1/2}$. Let

$\Delta_A Q_0(P) = \inf(\Delta Q_0(P))$ then $\Delta_A Q(P)$ corresponds to $\sup Q(x^1)$ and to the maximum range of x^1 in A .

3.2: Global Measurability and Completeness

Now let A be an open neighbourhood of a point m lying on a maximal integral curve Ω of X associated with P . In A the curve Ω is simply a coordinate curve along the coordinate x^1 . We require A to satisfy two additional conditions: (i) The arc $A \cap \Omega$ has a pre-assigned metric length d_Ω . (ii) The maximum range of x^1 (generally different from arc length) in A must equal the range of x^1 along the arc $A \cap \Omega$. We shall denote all such sets A along Ω by $\{A\}_\Omega$ and call $\{A\}_\Omega$ a reference class of local sets constructed over Ω . Note that d_Ω is the same for every $A \cap \Omega, A \in \{A\}_\Omega$. For different integral curves d_Ω may differ. We are now ready to introduce the notion of quantum global measurability.

Definition: A momentum P is quantum globally measurable iff, for every maximal integral curve Ω of the associated vector field X it is possible to elect a finite, upper bound $\Delta_\Omega Q_0(P)$ to the set of least uncertainties $\Delta_A Q_0(P)$ generated from a chosen reference class of local sets A constructed over Ω .

This definition is a natural generalization of that given in our paper 1, and enables by analogous arguments the deduction of the following theorem.

Theorem 2: On Completeness and Quantum-Global Measurability.

A momentum P is quantum globally measurable only if it is complete.

Proof: Apply the argument of paper 1 for a similar theorem.

Perhaps/

we should point out here that the notion of quantum-global measurability is applicable whether P is complete or not, since $Q_0(P)$ always exists, and that the least uncertainty $\Delta_A Q_0(P)$ in A corresponds to the maximum range Δx^1 of x^1 in A .

4: On the Relationship between Classical and Quantum Global Measurability.

4.1: Two illustrative examples on global measurability.

The first example is the familiar angular momentum L_z on the Euclidean plane \mathbb{R}^2 . In terms of global Cartesian coordinates (q^1, q^2) on \mathbb{R}^2 and the corresponding chart (q^1, q^2, p_1, p_2) on $T^*\mathbb{R}^2$, L_z assumes the familiar form $L_z = q^1 p_2 - q^2 p_1$. The associated complete vector field is $X = q^1 \partial / \partial q^2 - q^2 \partial / \partial q^1$, whose maximal integral curves are given in terms of the flow $\mathcal{E}_t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\mathcal{E}_t(q^1, q^2) = (q^1 \cos t - q^2 \sin t, q^1 \sin t + q^2 \cos t).$$

These integral curves form a family of concentric circles of centre $(0,0)$.

The canonical coordinates in terms of which X assumes the form $\partial / \partial \theta$ and the metric line element, $ds^2 = (dq^1)^2 + (dq^2)^2$ the form

$$dr^2 + r^2 d\theta^2$$

are the (almost global) plane polar coordinates (r, θ) defined in the usual manner.

We now determine whether L_z is classically globally measurable.

Let us choose, for a representative integral curve $\Omega_\alpha = \{(r, \theta) | r = \alpha > 0\}$, a family $\{B_\alpha\}_{\ell_\alpha}$ of local recoil sets B_α as the family of arcs B_α of Ω of length $\ell_\alpha \ll 2\pi\alpha$ centred at $r = \alpha, \theta = \alpha$.

Along each arc B_α of Ω we may introduce a local Cartesian

coordinate y^1 defined by $y^1 = \alpha\theta$ in terms of which $X = \alpha \partial / \partial y^1$

It is now immediate that the quantity $b(m), m \in \Omega_\alpha$, is simply

$$b(m) = \alpha, \quad \text{a constant independent of } m \in \Omega_\alpha. \text{ Equations}$$

(6) and (7) give the measured value of L_z to be $L_z = \alpha p_0^1$ with an error $\Delta_{B_\alpha} L_z = 0$. Consequently L_z is classically globally measurable. Furthermore it is possible to measure L_z without error within each recoil set B_α a result which reflects the rather special character of the angular momentum.

Let us now consider whether L_z is quantum globally measurable. Let A_0 be an open neighbourhood of m_0 on Ω_α defined by

$$A_0 = \{(r, \theta) | \alpha - \epsilon/2 < r < \alpha + \epsilon/2, \theta \in (0, 8\theta)\},$$

and let $A_\alpha, \alpha \in [0, 2\pi)$, be the set obtained by rotating A_0 through the angle α . We can choose $\{A_\alpha | \alpha \in [0, 2\pi)\}$ as a reference class $\{A_\alpha\}_{\Omega_\alpha}$ as defined in 3.2. An illustration of the situation is in figure 3. It is obvious by symmetry that

$\Delta_{A_\alpha} Q_0(L_z)$ is the same for every A_α because every $\phi_\alpha \in C_0^\infty(A_\alpha)$ corresponds to $\phi_{\alpha'} \in C_0^\infty(A_{\alpha'})$ and vice versa by rotation. It is also obvious that $\Delta_{A_\alpha} Q_0(L_z) = \inf(\Delta Q_0(L_z))$ is bounded; we conclude that L_z is quantum globally measurable. Notice that

$$\Delta_{A_\alpha} Q_0(L_z) \text{ is the same for every } A_\alpha.$$

As the second example we consider the momentum $P = (q^1 - q^2)p_1 + (q^1 + q^2)p_2$ again on the Euclidean plane \mathbb{R}^2 . The associated complete vector field is $X = (q^1 - q^2)\partial / \partial q^1 + (q^1 + q^2)\partial / \partial q^2$ whose maximal integral curves are given in terms of the flow-map $\mathcal{E}_t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\mathcal{E}_t(q^1, q^2) = e^t (q^1 \cos t - q^2 \sin t, q^1 \sin t + q^2 \cos t).$$

Geometrically these are spirals radiating from the critical point of at the origin. A set of coordinates (x^1, x^2) in terms of which X assumes the form $\partial / \partial x^1$ and the metric line-element ds^2

the form $ds^2 = 2 \exp(2(x^1+x^2))((dx^1)^2 + (dx^2)^2)$, may be defined using plane polar coordinates by

$$x^1 = \frac{1}{2}(\ln r + \theta), \quad x^2 = \frac{1}{2}(\ln r - \theta).$$

Some integral curves of X and a representative grid of coordinates (x^1, x^2) are displaced in figure 4. Given a maximal integral curve $\Omega_0 = \{(x^1, x^2) | x^2 = x_0^2\}$, we can introduce a parameter $S(x^1)$ by

$$S(x^1) = \int_{-\infty}^{x^1} 2^{1/2} \exp(x^1 + x_0^2) dx^1 = 2^{1/2} \exp(x^1 + x_0^2), \quad (14)$$

which measures the metrical arc-length from the critical point of X , to the point (x^1, x_0^2) on Ω_0 and in terms of which the quantity $b(m_0)$ along Ω_0 takes the value $b(m_0) = \partial S / \partial x^1 = 2^{1/2}$.

Returning to the question of classical global measurability of P , we choose as the family of local recoil sets the family $\{B_\alpha | \alpha \gg \frac{1}{2}d_0\}$ of arcs B_α of Ω_0 each of metric length d_0 and centred at $S=\alpha$ on Ω_0 . Now equations (6), (7) give the measured value of P expected in B_α to be $P = b(B_\alpha) p_0^1$ with error $\Delta_{B_\alpha} P = 6^{-1/2} d_0 |p_0^1|$, which is independent of α . Hence P is classically globally measurable.

Finally, addressing the problem of quantum global measurability of P , we construct on Ω_0 a reference class $\{A_\alpha | \alpha \gg \frac{1}{2}d_0\}$ of local sets A_α with the property that $A_\alpha \cap \Omega_0 = B_\alpha$. It follows from equation (14) that the range of x^1 within A_α is $\ln[(2\alpha+d_0)/(2\alpha-d_0)]$, and hence the least uncertainty is given by

$$\Delta_{A_\alpha} Q_0(P) \geq \frac{1}{2} \hbar / \ln[(2\alpha+d_0)/(2\alpha-d_0)],$$

which tends to infinity as α tends to infinity[†]. As a result P is not quantum globally measurable. This serves as a counter example to show that ^{classical} global measurability does not imply quantum global measurability. Furthermore, counter examples given in Appendix 1 show that quantum global measurability does not imply classical global measurability either, and the the converses of both Theorem 1 and Theorem 2 are false in general.

4.2: Exact Classical Global Measurability

Having observed that neither all complete, nor all classical globally measurable, momenta are quantum globally measurable, we now enquire if it is possible to identify a physically important subclass of the complete momenta which are both classically and quantum globally measurable. To this end we introduce a refinement of classical global measurability as follows.

Definition: A momentum P is exactly classically globally measurable (exactly measurable) iff there exists a family $\{B\}_{\Omega}$ of local recoil sets for every maximal integral curve Ω of the associated vector field with the property that the error Δ_B^P in every B is zero.

This concept is motivated by recognition of the unique character of these momenta in being globally measurable without error. A prime example of these is the angular momentum L_z studied in 4.1. The relevance of exact measurability lies in the following three theorems.

[†]: The approach to the critical point as $q^1 \rightarrow -\infty$ presents no problem since $\Delta_{A_\alpha} Q_0(P) \rightarrow 0$ as $\alpha \rightarrow \frac{1}{2}d_0$.

Theorem 3: (Appendix 2): Practical Criteria for Exact Measurability.

Let $P = \xi^i(q) p_i$ be a complete momentum with the associated vector field $X = \xi^i(q) \partial / \partial q^i$. Then P is exactly measurable iff either (1) for every maximal integral curve $\Omega = \Omega(s)$ parameterised by its arc length s , the quantity $b(m) = b(s)$ is constant along $\Omega(s)$ or
(2) $\xi^i(q)$ satisfy the equation

$$\xi^i \xi^j (\xi_{i;j} + \xi_{j;i}) = 0$$

in every chart q^i (the vertical bar denotes covariant differentiation).

Theorem 4: (Appendix 2): On exact Measurability and Quantum Global Measurability

If a momentum P is exactly measurable then it is quantum globally measurable.

Theorem 5: (Appendix 2): On Killing Momenta and Exact Measurability.

Every complete Killing momentum P defined by the requirement that its associated vector field X is Killing is exactly measurable and therefore quantum globally measurable.

These theorems demonstrate the cohesion and physical relevance of the concepts of global measurability because these notions apply to the physically important Killing momenta. These theorems also tell us the property of the Killing momenta concerning measurement. Quantum mechanically an exactly

measurable P has the further feature that the error $\Delta_A Q_0(P)$ depends only on the length d_0 of A and the particular integral curve Ω over which A is defined, but does not depend on the exact location of A on Ω . In particular a Killing momentum like L_z on \mathbb{R}^2 can be measured quantum mechanically along its integral curves with identical least error.

5. Conclusions:

We have generalised the concepts of global measurability and introduced the notion of exact measurability. We have shown the existence of a large and physically important class of momenta, i.e. the Killing momenta, which are complete, hence quantisable, and are exactly measurable, hence classically and quantum globally measurable. We have also established the following relationship among completeness and measurabilities:

$$\begin{aligned} \{\text{exact measurability}\} &\not\Rightarrow \{\text{classical global measurability}\} \\ \{\text{exact measurability}\} &\Rightarrow \{\text{quantum global measurability}\} \\ \{\text{completeness}\} &\Leftrightarrow \{\text{classical global measurability}\} \\ \{\text{completeness}\} &\Leftrightarrow \{\text{quantum global measurability}\} \\ \{\text{classical global measurability}\} &\not\Rightarrow \{\text{quantum global measurability}\} \end{aligned}$$

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Appendix 1: Counter Examples

(1): The failure of the converse of Theorems 1 and 2.

Consider, in the Euclidean space \mathbb{R} with global Cartesian chart y , the momentum

$$P = \xi(y) p_y, \quad \xi(y) = y^3 / (1 + y^2 \sin^2 y).$$

(i) P is complete. We have that

$$\int_{n\pi}^{(n+1)\pi} \xi^{-1}(y) dy = (n+\frac{1}{2}) / [n^2(n+1)^2 \pi^2] + \int_{n\pi}^{(n+1)\pi} y^{-1} \sin^2 y dy$$

$$\geq (n+\frac{1}{2}) / [n^2(n+1)^2 \pi^2] + (n+1)^{-1} \pi^{-1} \int_{n\pi}^{(n+1)\pi} \sin^2 y dy = (n+\frac{1}{2}) / [n^2(n+1)^2 \pi^2] + \frac{1}{2} (n+1)^{-1}.$$

$$\text{Hence, } \forall y_0 > 0 \quad \int_{y_0}^y \xi^{-1}(y) dy \mapsto \infty \quad \text{as } y \mapsto \infty.$$

Also $\int_{y_0}^y \xi^{-1}(y) dy \mapsto -\infty$ as $y \mapsto 0$. Similar results hold for $y_0 < 0$; hence P is complete.

(ii) Quantum global measurability.

For any interval $A = (y, y+d)$ of fixed length d , the range within A of coordinate x in which $\xi(y) d/dy$ goes to d/dx is

$$\Delta x = \int_y^{y+d} \xi^{-1}(y) dy, \quad y > 0.$$

This is easily seen to tend to zero as $y \mapsto \infty$. Hence $\Delta_Q(P)$ is unbounded and P fails to be quantum globally measurable.

(iii) Classical global measurability

It will suffice to show that the errors $\Delta_B b$ generated by the class of sets $B_n = (n\pi, n\pi+d)$ for every fixed $d > 0$ increase without limit as n tends to infinity. Set $B_n = (\alpha, \beta)$ and consider the mean $b(B_n)$ we have that

$$b(B_n) = d^{-1} \int_{\alpha}^{\beta} \xi(y) dy \leq d^{-1} \beta^3 \int_0^d (1 + \alpha^2 \sin^2 y)^{-1} dy.$$

Therefore, upon noting the integrals

$$\int_0^{d < \pi/2} (1 + \alpha^2 \sin^2 y)^{-1} dy = (1 + \alpha^2)^{-1/2} \tan^{-1}((1 + \alpha^2)^{1/2} \tan d),$$

$$\int_0^{d < \pi/2} (1 + \alpha^2 \cos^2 y)^{-1} dy = (1 + \alpha^2)^{-1/2} \tan^{-1}((1 + \alpha^2)^{1/2} \tan d),$$

we may deduce the upper bound

$$b(B_n) \leq \frac{1}{2} \beta^3 d^{-1} ([2d/\pi] + \frac{1}{2}) \pi (1 + \alpha^2)^{-1/2},$$

in which $[2d/\pi]$ denotes the integer part of $2d/\pi$. To obtain a lower bound to the corresponding errors $\Delta_B b$, we consider the integral

$$\eta(B_n) = d^{-1} \int_{\alpha}^{\beta} \xi^2(y) dy \geq d^{-1} \alpha^6 \int_0^d (1 + \beta^2 \sin^2 y)^{-2} dy,$$

which together with the identity

$$\int_0^{d < \pi/2} (1 + \beta^2 \sin^2 y)^{-2} dy = \frac{(1 + \beta^2/2) \tan^{-1}((1 + \beta^2)^{1/2} \tan d) + \frac{\beta^2 \sin d \cos d}{2(1 + \beta^2)(1 + \beta^2 \sin^2 d)}}{(1 + \beta^2)^{3/2}},$$

yields the bounds

$$\eta(B_n) \geq \begin{cases} \frac{1}{2} d^{-1} \alpha^6 (1 + \beta^2/2) (1 + \beta^2)^{-3/2} \pi, & d \geq \pi/2 \\ d^{-1} \alpha^6 (1 + \beta^2/2) (1 + \beta^2)^{-3/2} \tan^{-1}((1 + \beta^2)^{1/2} \tan d), & d < \pi/2 \end{cases}$$

Finally observe that for sufficiently large β or equivalently sufficiently large n , $(1+\beta^2)^{1/2} \tan d > 1$, and that therefore in all cases we have that, for sufficiently large $n(d)$

$$\eta(B_n) \geq \frac{1}{4} d^{-1} \alpha^6 (1+\beta^2/2) (1+\beta^2)^{-7/2} \pi.$$

Finally, since $\Delta_{B_n} b = \eta(B_n) - b^2(B_n)$ we obtain the asymptotic lower bound $\Delta_{B_n} b \sim \pi \alpha^5 / 8d \sim \infty$, as was required.

(2) Quantum and classical global measurability.

Consider, again in \mathbb{R} the momentum $P = \xi(y) p_y$, $\xi(y) = y^2 / (1+y^4 \sin^2 y)$. using procedure similar to (1) above one can readily verify that this

P is complete and quantum globally measurable, but is not classically globally measurable.

Appendix 2 On Exact Measurability

(1) Proof of Theorem 3.

Part one is obviously true. So only part two needs proving.

Let x^i be a chart in terms of which $X = \xi^i \partial / \partial x^i = \partial / \partial x^1$.

Let the metric in charts q^i and x^i be denoted respectively by

$g_{ij}(q)$ and $\bar{g}_{ij}(x)$. Then $b^2(m) = \bar{g}_{11}(x)$ (see Appendix 4 of paper 1), or $b^2(m) = g_{ij}(\partial q^i / \partial x^1)(\partial q^j / \partial x^1) = g_{ij} \xi^i \xi^j$ (evaluated at $m \in \Omega$). Evaluating at $m \in \Omega$ and along Ω we have

$$\partial b / \partial s = 0 \iff \partial b / \partial x^1 = 0 \iff \xi^k \partial / \partial q^k (g_{ij} \xi^i \xi^j) = 0,$$

the last expression leading directly to the required result.

(2) Proof of Theorem 4.

P being exactly measurable implies that $b(s) = b_0$, a constant by Theorem 3. Using $b = \partial s / \partial x^1$, we have $dx^1 = b_0^{-1} ds$ which, after integrating, gives the maximum range $\Delta x^1 = b_0^{-1} \Delta s$. Since $\Delta s = d_0$ is the same for every local set A , Δx^1 is also the same for every A . Consequently $\Delta_A Q_0(P)$ the least uncertainty corresponding to the maximum range Δx^1 is the same for every A resulting in the quantum global measurability of P . Notice also that $\Delta_A Q_0(P)$ is the same for every A .

(3) Proof of Theorem 5.

This follows from Theorem 3 since a Killing momentum $P = \xi^i p_i$ satisfies the Killing equations $\xi_{i;j} + \xi_{j;i} = 0$.

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Captions

Figure 1: The geometry of the test/reference particle collision within the set A .

The trajectory of the test particle of Cartesian momenta p_i^0 is the arc $abcd$, the recoil set B demarking the extent of the uncertainty in position of the test particle during the collision; and the trajectory of the reference particle, aligned with the local flow direction efc and bearing Cartesian momentum p_i^0 is similarly $efgh$, the local set B' marking the recoil displacement of the reference particle during the collision.

Figure 2: To illustrate the local recoil sets B_α defined on the integral curves of a vector field.

The underlying graph is a family of maximal integral curves of the vector field $X = q^1(\frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2})$ which has a set of critical points $q^1 = 0$ which separates the flow into two distinct regions $q^1 \geq 0$. We show along typical flow-lines of the vector field a representative group of local recoil sets B_α together with the local Cartesian coordinate directions.

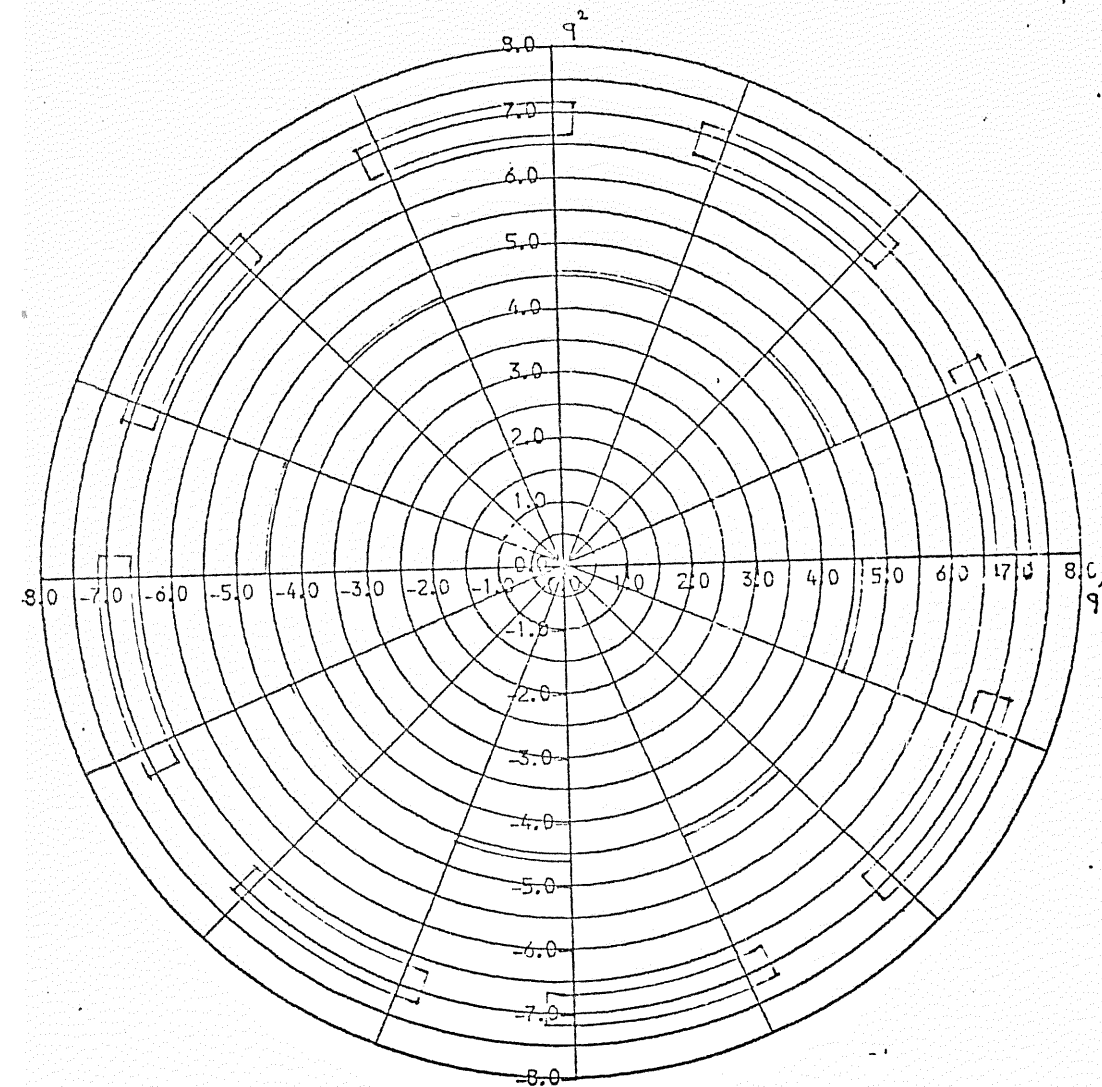
Figure 3: To illustrate the maximal integral curves of L_x and of a representative grid of the polar coordinate system (r, θ) .

We also illustrate on the figure some typical local recoil sets B_α and some local sets A_α .

Figure 4: To illustrate the maximal integral curves of the vector field $X = (q^1 - q^2) \frac{\partial}{\partial q^1} + (q^1 + q^2) \frac{\partial}{\partial q^2}$.

The maximal integral curves of X are the marked counterclockwise spirals emanating from the origin of coordinates, the second family of curves being a representative set of coordinate arcs perpendicular to the integral curves of X .

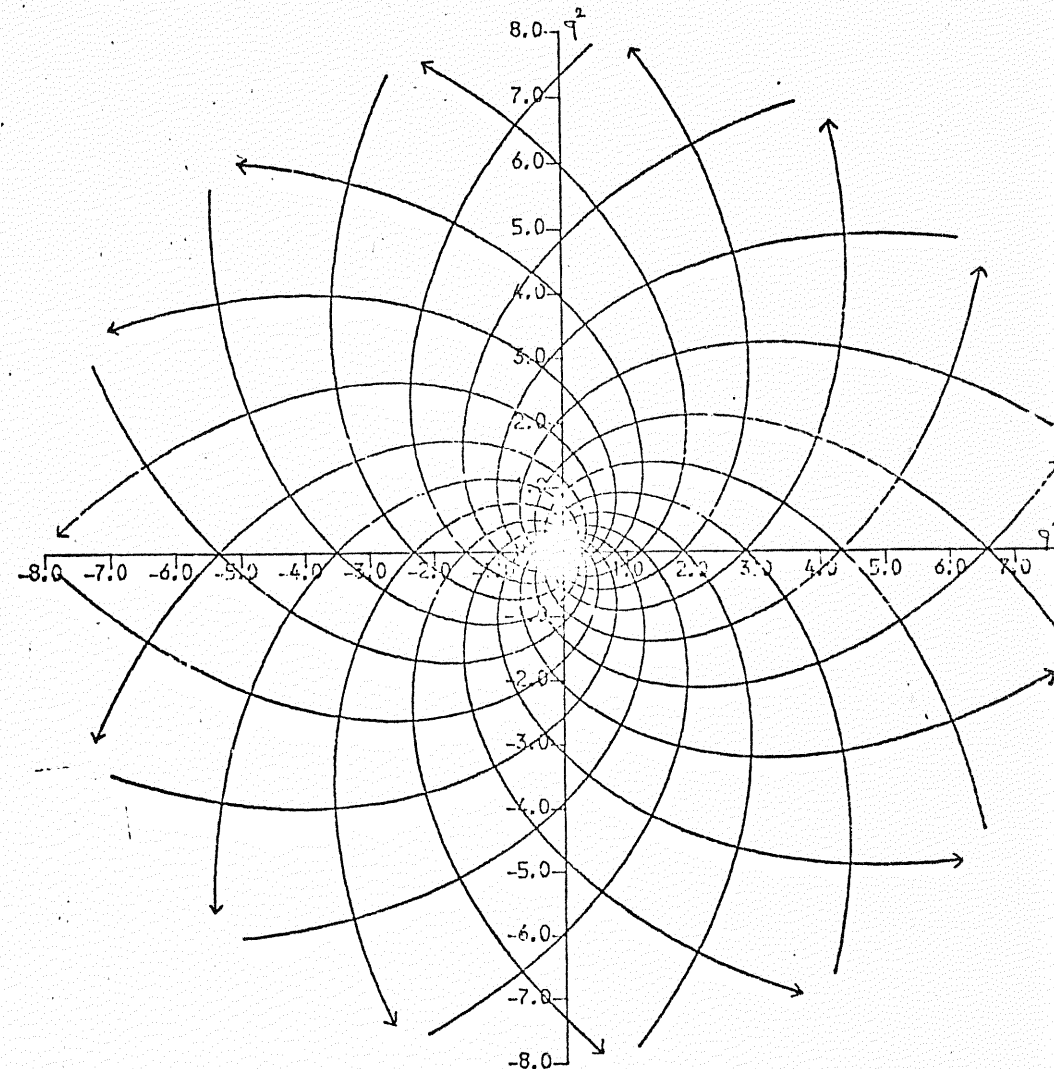
Figure 3: To illustrate the maximal integral curves of L_z and a representative grid of the polar coordinate system (r, θ) .



We also illustrate on the figure some typical local recoil sets B_α and some local sets A_α .

Figure 4: To illustrate the maximal integral curves of the vector field

$$X = (q^1 - q^2) \partial / \partial q^1 + (q^1 + q^2) \partial / \partial q^2.$$



The maximal integral curves of X are the marked counterclockwise spirals emanating from the origin of coordinates, the second family being a representative set of coordinate surfaces perpendicular to the integral curves of X .

List of Hand-written Symbols

<u>Symbol</u>	<u>Interpretation</u>	<u>Symbol</u>	<u>Interpretation</u>
A	Bold capital A	a	lower case a
B	Bold capital B	b	lower case b
C	Bold capital C	d	lower case d
D	Bold capital D	e	lower case e
E	Bold capital E	g	lower case g
G	Bold capital G	i	lower case i
H	Bold capital H	j	lower case j
K	Bold capital K	k	lower case k
L	Bold capital L	l	lower case l
M	Bold capital M	m	lower case m
P	Bold capital P	n	lower case n
Q	Bold capital Q	p	lower case p
T	Bold capital T	q	lower case q
X	Bold capital X	r	lower case r
		s	lower case s
		t	lower case t
		x	lower case x
		y	lower case y
		z	lower case z

<u>Symbol</u>	<u>Interpretation</u>	<u>Symbol</u>	<u>Interpretation</u>
Δ	Greek capital "delta"	α	Greek lower case "alpha"
Ω	Greek capital "omega"	β	Greek lower case "beta"
Φ	Greek capital "phi"	γ	Greek lower case "gamma"
		δ	Greek lower case "delta"
		ϵ	Greek lower case "epsilon"
		η	Greek lower case "eta"
		θ	Greek lower case "theta"
		ξ	Greek lower case "xi"
		π	Greek lower case "pi"
		σ	Greek lower case "sigma"
		χ	Greek lower case "chi"
		ψ	Greek lower case "psi"
		ϕ	Greek lower case "phi"
		ω	Greek lower case "omega"

List of Hand-written Symbols

<u>Symbol</u>	<u>Interpretation</u>
'	raised dash
-	raised horizontal bar
<	less than / equal to
>	greater than / equal to
\int	integral sign
	modulus bars
{,}	curly brackets
∞	infinity symbol
o	subscript zero
†	dagger (raised)
\hbar	Dirac's constant (lower case H with cross piece)
\cap	small intersection symbol
\mathbb{R}	standard symbol for the real numbers
\rightarrow	tends to
\Rightarrow	implies
\forall	universal quantifier (inverted capital A)